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Stochastic Modelling of Fractures in Rock Masses

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The Problem

3D Rock Fractures and Fracture Networks
A few Examples
One Simple Application
Some examples of fracture trace images taken from rock exposure surfaces
Stochastic Approach:
Marked Point Process
Point representation of fractures
Model Construction

Non-parametric approach

Parametric approach

• Maximum likelihood estimation

\[ l(\theta, A) = \left\{ \prod_{i=1}^{n} \lambda^{\theta}(X_{i}) \right\} \cdot e^{-\int_{A}^{\lambda^{\theta}(u)v(du)}} \]

• Least square estimation

\[ D(\theta) = \int_{t_{0}}^{t_{0}} \left[ [\hat{K}(t)]^{c} - [K(t, \theta)]^{c} \right]^{2} dt \]
Non-parametric Estimation of Point Intensity

Common approach – Kernel estimation

\[
\lambda_h(X) = \frac{1}{p_h(X)} \sum_{i=1}^{N} \frac{1}{h^2} k\left(\frac{X - X_i}{h}\right) \quad \text{where} \quad p_h(X) = \frac{1}{h^2} \int_{\mathbb{R}} k\left(\frac{X - w}{h}\right) \cdot dw
\]

Epanechnikov kernel

\[
k(x, y) = \begin{cases} 
\frac{9}{16} \left[1 - \left(\frac{x}{h}\right)^2\right] \cdot \left[1 - \left(\frac{y}{h}\right)^2\right] & \text{if} \ |x| < h \text{ and } |y| < h, \\
0 & \text{otherwise}.
\end{cases}
\]

\(h\) – smoothing bandwidth
Bandwidth Selection

Common approach – Cross-validation

\[ CV(h) = \int_{\mathbb{R}} \hat{\lambda}_h(X) \cdot dX - \frac{2}{N} \sum_{i=1}^{N} \hat{\lambda}_h^{-i}(X_i) \]
Bandwidth selection based on cross-validation technique

(a) Cluster points

(b) CV vs BW

NP model with BW=0.05

Non-homogeneous points

CV vs BW

Cluster realisation with \( \sigma^2 = 0.1 \)

CV for different \( \sigma^2 \):
- \( \sigma^2 = 0.25 \)
- \( \sigma^2 = 0.10 \)
- \( \sigma^2 = 0.05 \)
- \( \sigma^2 = 0.025 \)
- \( \sigma^2 = 0.015 \)
Bandwidth Selection

Our approach – Multi-objective optimisation based on summary statistics

\[ \varepsilon(h) = \int_T [S - S_h]^2 \cdot dt \]

\[
\begin{align*}
\varepsilon_a(h) &= w_1 \cdot \varepsilon_1(h) + w_2 \cdot \varepsilon_2(h) + w_3 \cdot \varepsilon_3(h) + \cdots \\
\text{arg} \ h_{opt} &= \text{minimum} \left( \varepsilon_a(h) \right)
\end{align*}
\]
Apply the technique to cluster processes
Apply the technique to inhomogeneous processes
Multi-objective bandwidth optimisation for the Example data set based on summary statistics

(a) $\varepsilon(h)$ based on $K$-function

(b) $\varepsilon(h)$ based on inter-event distance

(c) $\varepsilon(h)$ based on nearest-event distance

(d) Average error function $\varepsilon_a(h)$
Simulation and model assessment

(a) Final model of the dataset
(b) One realisation of the model
(c) $\hat{h}(t)$ of the data and the model
(d) $\hat{K}(t)$ of the data and the model
Simulation of fractures and fracture network

- Poisson point process model
- Models for length and orientation
- Models for other marks associated with fractures

Procedures of simulating fractures and fracture networks
Simulated patterns

Simulated fracture traces for example data set 1
Conclusions

1. Limitation of traditional cross-validation technique
2. Flexibility of the multi-objective optimisation method
3. Effectiveness of different summary statistics
4. Suitability for rock fracture modelling
That’s all, folks!
Non-parametric Density Model

Optimal non-parametric model
Parametric approach – inhomogeneous model

$$\lambda(x, y) = \begin{cases} 
    a + b \cdot e^{\frac{-(x-13.4)^2 + (y-17.3)^2}{c + d}} + u \cdot e^{\frac{-(x-16.4)^2 + (y-13.4)^2}{v + w}} & (x, y) \in \mathbb{R}' \\
    0 & \text{otherwise}
\end{cases}$$

Marginal optimisation

$$\hat{p}_i = \text{maximum} \{ l(p_i / [p_j = \hat{p}_j, j=1,2,..., j \neq i]) \}$$

where $p_i$ and $p_j$ refer to parameters $a, b, c, d, u, v$ or $w$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.62</td>
<td>8.1</td>
<td>4.1</td>
<td>0.45</td>
<td>4.2</td>
<td>2.5</td>
<td>1.6</td>
</tr>
</tbody>
</table>

Table 1 Estimated parameters for the proposed model using ML method
Parametric approach – cluster model

Fitting cluster parameters for the fracture point dataset
Parametric approach – Gaussian Cox model

\[ C(x, y) = \begin{cases} \mathcal{N}(0.0, 0.4) & (x, y) \in \mathbb{R}' \\ 0 & \text{otherwise} \end{cases} \]
Model Verifications

(a) Inter-event distance

(b) Nearest-event distance

(c) K-function

① - non-homogeneous model
② - cluster model
③ - Cox model
Summary and conclusions

• The help of non-parametric model

• The non-homogeneous model gives the best performance for this dataset

• Limitation of cluster model

• Local scale modelling by Cox process
Current Issues

- Correlations between mark and location
- Correlations between marks
- Higher dimension point processes

Procedures of simulating fractures and fracture networks

- Extend to 3D space
Model Verifications – Non-parametric model

Fig. 6 (a) Inter-event distance

Fig. 6 (b) K-function

Fig. 6 (c) Nearest event distance

Fig. 7 A set of simulated fractures