1. Introduction
The essential problem is to model the geometry of fractures from sparse data. On the basis of geometry a fracture can be modelled as a point or line process in the plane or, in three dimensions, as a random closed set. Although our work will be extended to the three-dimensional case, we limit this presentation to planar models. We model location as a Poisson point process in which the points are the midpoints of line representations of fractures. Point pattern analysis and model estimation can then be conducted on the point set. As points do not provide any information about fracture characteristics, such as length and orientation, we assign marks (or weights) to the points to represent these characteristics. The inclusion of marks provides a marked point data set that can be modelled as a Marked Poisson process.

Once the models for points and marks are established they can be used to simulate points and associated marks for other regions that are assumed to have the same, or similar, characteristics as the point dataset from which the models are derived. Fracture sets are simulated by replacing the simulated marks with the properties they represent. Examples of marked characteristics of fractures are length, orientation and aperture in 2D fracture traces and size, shape, orientation and aperture profile in 3D fractures.

2. Point representation of a fracture dataset
Figure 1 provides an example of a fracture data set taken from a rock exposure. The fractures are represented by straight lines with location defined by the midpoints of these lines. Figure 2 shows the point data set comprising the line midpoints of the fractures in Figure 1.

If marks are used to represent trace length and orientation then each point has two marks. For more accurate modelling, fractures can be grouped together into sub-sets according to their geological characteristics and the sub-sets can then be separately modelled. Correlations between sub-sets can be built into the model to take account of inter-set relationships (e.g., existing fractures will influence subsequent fracture propagation).
3. Point pattern analysis
The first step in modelling spatial point patterns is pattern analysis. Three methods are commonly used and brief descriptions follow.

3.1 Distance method
The three measures used in this analysis are the inter-event distance, $t$, the nearest event distance, $y$, and the point to nearest event distance, $x$, are the three main measures for this analysis. These measures are shown in Figure 3. For the case of a homogeneous Poisson process the distributions of these measures are given in Equation 1. The distance analyses of $h(t)$ and $g(y)$ for the dataset shown in Figure 1 are given in Figure 4. Note the considerable departure of this dataset from a homogeneous Poisson model.

\[
\begin{align*}
 h(t) &= \begin{cases} 
 \frac{2pt}{L^2} - \frac{8t^2}{L^2} + \frac{2t^3}{L^2} & (t^2 \leq L^2) \\
 4t \cdot \arcsin\left(\frac{2L^2}{t^2} - 1\right) & (L^2 < t^2 \leq 2L^2) 
\end{cases} \\
 g(y) &= 2\rho \cdot y \cdot e^{-\gamma y^2} \\
 f(x) &= 2\rho \cdot x \cdot e^{-\gamma x^2}
\end{align*}
\]

Figure 3 Definitions of distance measures

Figure 4 Distance analysis of the example
3.2 Quadrat count analysis
Quadrat count analysis is a type of point number variance analysis. It calculates an index of dispersion (IOD), which is an indication of the degree of departure from a homogeneous pattern. The IOD is calculated as:

\[
IOD = \sum_{i=1}^{m} \frac{(n_i - \bar{n})^2}{(m - 1) \cdot \bar{n}}
\]  

(2)

where \(n_i\) is the number of points in quadrat \(i\) and \(m\) is the number of quadrats. The quadrats used for the analysis can be either random or regular, as shown in Figure 5. An example of the quadrat count analysis for the dataset given in Figure 1 is shown in Figure 6.

![Random quadrats and contiguous quadrats](image)

Figure 5 Quadrats used for quadrat count analysis

3.3 Second moment analysis – K function
The K-function is by far the most powerful tool for revealing spatial correlations of point patterns. It is defined as:

\[
K(t) = E \left[ \text{number of further events within distance } t \text{ of an arbitrary event} \right]
\]

(3)

where \(\lambda\) is the density function. The relationship between the K-function and the second moment density function \(\lambda_2\) is given by:

\[
\begin{align*}
\lambda \cdot K(t) &= \frac{2\pi}{\lambda} \int_0^t \lambda_2(x) \cdot x \cdot dx \quad \text{or}, \\
\lambda_2(t) &= \frac{\lambda^2}{2\pi t} \cdot K'(t)
\end{align*}
\]

(4)

and the covariance density is then calculated as:

\[
\gamma(t) = \lambda_2(t) - \lambda^2
\]

(5)

Figure 7 shows the K-function analysis for the dataset in Figure 1 and again demonstrates a significant departure from a homogeneous pattern.
4. Point pattern modelling

Once a pattern analysis has been conducted and a feasible model has been identified, the parameters of the model can be derived by, for example, maximum likelihood or least squares. The maximum likelihood estimator $\hat{\theta}$ of the parameter $\theta$ is the estimator that maximises the likelihood function $I(\theta, A)$, with respect to $\theta$:

$$I(\theta, A) = \left\{ \prod_{i=1}^{n} \lambda^\theta(X_i) \right\} \cdot e^{-\lambda^\theta \int \lambda(du)} \quad (6)$$

The least squares estimator $\hat{\theta}$ is the estimator that minimizes $D(\theta)$:

$$D(\theta) = \int_0^T \left[ (K(t))^c - (K(t, \theta))^c \right]^p \, dt \quad (7)$$

Non-parametric approaches can also be used by, for example, using a kernel estimate or a distance (nearest neighbour) estimate. For example, the non-parametric model for the example dataset in Figure 1 is shown in Figure 8. The output from a point simulation based on this model is given in Figure 9 and is clearly very similar to the point data in Figure 2. The similarities between Figures 2 and 9 can be confirmed in a more objective manner by a pattern analysis of the simulation.
Once the point simulation is verified, the marks for each of the simulated points can be simulated. Converting the simulated marks to their corresponding properties yields a simulated fracture set such as that shown in Figure 10.

5. Other applications
Although the applications described here are for the characterisation of rock masses they are also relevant to many other areas including materials science (e.g. defects in metal surfaces) and microscopy.

6. References
Moller, J., Syversveen, A. & Waagepetersen, R., Log Gaussian Cox processes, internal report, Aalborg University, Germany.

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